Problem Set 15 Solutions

Problem 15.1

(a) Estimate the diamagnetic susceptibility of a typical solid. (b) Using this, estimate the field strength needed to levitate a frog, assuming a gradient that drops to zero across the frog. Express your answer in teslas.

Solution: (a) The diamagnetic susceptibility is given by Equation (14.15) from the page 234 :

$$\chi_m = -\mu_0 \frac{q^2 Z r^2}{4 m_e V}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of free space, $q = e = 1.602 \times 10^{-19}$ C is the electron charge, $m_e = 9.11 \times 10^{-31}$ kg is the electron mass, Z is the atomic number, r is the atomic radius, and V is the atomic volume.

Let's estimate typical values for a solid:

- Atomic number $Z \approx 20$
- Atomic radius $r \approx 1 \times 10^{-10}$ m
- Atomic volume $V \approx 1.5 \times 10^{-29}$ m^3 (e.g., based on typical solid density ~ 5000 kg/m^3 and atomic mass ~ 50 amu)

Plugging these values in:

$$\chi_m \approx -(4\pi \times 10^{-7}\,\text{H/m}) \frac{(1.602 \times 10^{-19}\,\text{C})^2 \times 20 \times (1 \times 10^{-10}\,\text{m})^2}{4 \times (9.11 \times 10^{-31}\,\text{kg}) \times (1.5 \times 10^{-29}\,\text{m}^3)}$$

$$\chi_m \approx -\frac{(1.257 \times 10^{-6}) \times (2.566 \times 10^{-38}) \times 20 \times (10^{-20})}{(4) \times (9.11 \times 10^{-31}) \times (1.5 \times 10^{-29})}$$

$$\chi_m \approx -\frac{6.45 \times 10^{-63}}{5.47 \times 10^{-59}} \approx -1.18 \times 10^{-4}$$

This estimate is somewhat larger than typical values (e.g., Au is -3.4×10^{-5}, water is -9.05×10^{-6}). The simplified model and estimated parameters contribute to the difference. Let's use a typical value $\chi_m \approx -1 \times 10^{-5}$ for estimations.

(b) To levitate a frog, the magnetic force must balance gravity. The force on a material with volume V_{frog} in a magnetic field gradient is given by Equation (14.7) :

$$F_{mag} = -V_{frog} \mu_0 \chi_m H \frac{dH}{dz}$$

Using $B = \mu_0(H + M) \approx \mu_0 H$ for diamagnetic materials (since $M = \chi_m H$ and $|\chi_m| \ll 1$), we have $H \approx B/\mu_0$. The force becomes:

$$F_{mag} = -V_{frog} \mu_0 \chi_m \frac{B}{\mu_0} \frac{d(B/\mu_0)}{dz} = -V_{frog} \frac{\chi_m}{\mu_0} B \frac{dB}{dz}$$

For levitation, $F_{mag} = F_{gravity} = m_{frog} g = \rho_{frog} V_{frog} g$.

$$-V_{frog} \frac{\chi_m}{\mu_0} B \frac{dB}{dz} = \rho_{frog} V_{frog} g$$

Since χ_m is negative for diamagnetic materials, the force is directed opposite to the gradient, i.e., upwards if the field strength decreases upwards.

$$-\frac{\chi_m}{\mu_0} B \frac{dB}{dz} = \rho_{frog} g$$

We are told the gradient drops to zero across the frog. Let the size of the frog be L. We can approximate the gradient as $\frac{dB}{dz} \approx \frac{B}{L}$.

$$-\frac{\chi_m}{\mu_0} B \frac{B}{L} = \rho_{frog} g \quad \Longrightarrow \quad B^2 = -\frac{\mu_0 L \rho_{frog} g}{\chi_m}$$

Let's use the susceptibility of water for the frog: $\chi_m \approx -9.05 \times 10^{-6}$. Assume frog density $\rho_{frog} \approx 1000$ kg/m^3 and size $L \approx 0.1$ m.

$$B^2 = -\frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A}) \times (0.1\,\mathrm{m}) \times (1000\,\mathrm{kg/m^3}) \times (9.8\,\mathrm{m/s^2})}{-9.05 \times 10^{-6}}$$

$$B^2 \approx \frac{1.23 \times 10^{-3}}{9.05 \times 10^{-6}} \approx 136\,\mathrm{T}^2$$

$$B \approx \sqrt{136}\,\mathrm{T} \approx 11.7\,\mathrm{T}$$

This is a very strong magnetic field, achievable with specialized magnets. The famous frog levitation experiment at Radboud University used about 16 T.

Problem 15.2

Estimate the size of the direct magnetic interaction energy between two adjacent free electrons in a solid, and compare this to size of their electrostatic interaction energy. Remember that the field of a magnetic dipole m is given by Eq. (14.42).

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{x}(\hat{x}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}\right] \qquad (14.42)$$

Solution: The magnetic moment of a free electron is the Bohr magneton, $m = \mu_B = 9.274 \times 10^{-24}$ J/T (Eq. 14.19). Let two electrons be separated by a typical interatomic distance $r \approx 0.2\,\mathrm{nm} = 2 \times 10^{-10}$ m.

Magnetic Interaction Energy: The energy of a magnetic dipole $\vec{m}_2$ in the magnetic field $\vec{B}_1$ created by dipole $\vec{m}_1$ is $E_{mag} = -\vec{m}_2 \cdot \vec{B}_1$. Let's assume the electron spins (magnetic moments) are parallel and aligned along the z-axis, $\vec{m}_1 = \vec{m}_2 = \mu_B sp\hat{z}$. Let the separation vector also be along the z-axis, $\vec{x} = r\hat{z}$, so $\hat{x} = \hat{z}$. Using Eq. (14.42):

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\left[\frac{3\hat{z}(\hat{z}\cdot(\mu_B\hat{z})) - \mu_B\hat{z}}{r^3}\right] = \frac{\mu_0}{4\pi}\left[\frac{3\hat{z}(\mu_B) - \mu_B\hat{z}}{r^3}\right] = \frac{\mu_0}{4\pi}\frac{2\mu_B\hat{z}}{r^3}$$

The interaction energy is:

$$E_{mag} = -(\mu_B\hat{z}) \cdot \left(\frac{\mu_0}{4\pi}\frac{2\mu_B\hat{z}}{r^3}\right) = -\frac{\mu_0\mu_B^2}{2\pi r^3}$$

$$E_{mag} = -\frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A}) \times (9.274 \times 10^{-24}\,\mathrm{J/T})^2}{2\pi \times (2 \times 10^{-10}\,\mathrm{m})^3}$$

$$E_{mag} = -\frac{(2 \times 10^{-7}) \times (8.60 \times 10^{-47})}{(8 \times 10^{-30})} = -\frac{1.72 \times 10^{-53}}{8 \times 10^{-30}} \approx -2.15 \times 10^{-24}\,\mathrm{J}$$

Converting to eV: $E_{mag} \approx -2.15 \times 10^{-24}\,\mathrm{J}/(1.602 \times 10^{-19}\,\mathrm{J/eV}) \approx -1.3 \times 10^{-5}$ eV.

Electrostatic Interaction Energy: The electrostatic potential energy between two electrons is:

$$E_{elec} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

where $\epsilon_0 = 8.854 \times 10^{-12}$ F/m is the permittivity of free space and $e = 1.602 \times 10^{-19}$ C is the electron charge.

$$E_{elec} = \frac{1}{4\pi(8.854 \times 10^{-12}\,\mathrm{F/m})}\frac{(1.602 \times 10^{-19}\,\mathrm{C})^2}{2 \times 10^{-10}\,\mathrm{m}}$$

$$E_{elec} = (8.988 \times 10^9\,\mathrm{N \cdot m^2/C^2})\frac{2.566 \times 10^{-38}\,\mathrm{C}^2}{2 \times 10^{-10}\,\mathrm{m}} \approx 1.15 \times 10^{-18}\,\mathrm{J}$$

Converting to eV: $E_{elec} \approx 1.15 \times 10^{-18}\,\mathrm{J}/(1.602 \times 10^{-19}\,\mathrm{J/eV}) \approx 7.2$ eV.

Comparison: The ratio of the magnitudes is:

$$\frac{|E_{mag}|}{|E_{elec}|} = \frac{2.15 \times 10^{-24}\,\mathrm{J}}{1.15 \times 10^{-18}\,\mathrm{J}} \approx 1.9 \times 10^{-6}$$

The direct magnetic interaction energy between adjacent electrons is about six orders of magnitude smaller than their electrostatic interaction energy. This supports the statement in the text (page 237) that magnetic forces are typically much smaller than electrostatic interactions and that the latter are responsible for phenomena like ferromagnetism (via the exchange interaction).

Problem 15.3

Using the equation for the energy in a magnetic field, describe why: (a) A permanent magnet is attracted to an unmagnetized ferromagnet. (b) The opposite poles of permanent magnets attract each other.

Solution: Systems tend to minimize their potential energy. The energy associated with magnetic fields can be considered. One relevant expression is the energy stored in the magnetic field, $E = \frac{1}{2\mu}\int B^2 dV$ (Eq. 14.33), or the energy density $U = \frac{1}{2}\vec{B}\cdot\vec{H}$ (page 231). Another view is the potential energy change when introducing a material into a field, $\Delta E = \frac{1}{2}V\mu_0\chi_m H^2$ (page 2[Eq. 14.6), leading to a force $F = -\nabla(\Delta E)$.

(a) **Permanent magnet attracting an unmagnetized ferromagnet:** A ferromagnet has a very high relative permeability $\mu_r \gg 1$, meaning a large positive magnetic susceptibility $\chi_m = \mu_r - 1 > 0$. When an unmagnetized ferromagnet is placed near a permanent magnet, the external field $\vec{H}_{ext}$ from the permanent magnet magnetizes the ferromagnet, inducing a magnetization $\vec{M}$ within it, largely parallel to $\vec{H}_{ext}$. The potential energy change upon introducing the material is $\Delta E = \frac{1}{2}V\mu_0\chi_m H^2$[cite: 17]. Since $\chi_m > 0$ for a ferromagnet, the energy is lower (ΔE is more negative if we consider the energy difference relative to vacuum) when H is larger. The system will move to minimize this energy, meaning the ferromagnet will move towards regions of stronger field H, i.e., towards the permanent magnet. Alternatively, using the force equation $F = -V\mu_0\chi_m H\frac{dH}{dz}$ (Eq. 14.7), since $\chi_m > 0$, the force is in the direction of increasing field strength ($H\frac{dH}{dz} > 0$). This means the ferromagnet is pulled towards the permanent magnet where the field is stronger. Also, considering the field energy $E = \int \frac{1}{2}\vec{B}\cdot\vec{H}dV$[cite: 10], the presence of a high permeability material ($\mu \gg \mu_0$) concentrates the magnetic field lines. The total energy is minimized when the high-μ material occupies the region where the field created by the permanent magnet is strongest, leading to an attractive force.

(b) **Opposite poles of permanent magnets attract:** Consider two permanent magnets as dipoles $\vec{m}_1$ and $\vec{m}_2$. The potential energy of interaction is $E = -\vec{m}_2\cdot\vec{B}_1$, where $\vec{B}_1$ is the field produced by magnet 1 at the location of magnet 2. Magnetic field lines point away from North poles and towards South poles. If the North pole of magnet 1 faces the South pole of magnet 2:

- The field $\vec{B}_1$ near the North pole points away from magnet 1.
- The dipole moment $\vec{m}_2$ associated with magnet 2 points from its South pole towards its North pole (internal convention) or represents the pole strength orientation (external field perspective). Let's consider the force perspective: the South pole is attracted to the North pole.
- Alternatively, using potential energy: Let $\vec{m}_1$ point along $+z$ (North pole up) and $\vec{m}_2$ point along $-z$ (South pole up, North pole down). The field $\vec{B}_1$ above the North pole points generally along $+z$. The dipole $\vec{m}_2$ points along $-z$. The energy $E = -\vec{m}_2\cdot\vec{B}_1 = -(-\mu_B\hat{z})\cdot(B_1\hat{z}) = \mu_B B_1$. This seems wrong, suggesting repulsion.

Let's re-evaluate $E = -\vec{m}\cdot\vec{B}$. Consider the interaction energy between poles. A simpler view: The field lines originating from the North pole of magnet 1 enter the South pole of magnet 2. The configuration where the field lines flow easily between the magnets corresponds to lower field energy in the surrounding space compared to when like poles face each other, where field lines must curve sharply and occupy more volume. The system seeks to minimize the stored field energy, which occurs when opposite poles are brought together, resulting in attraction. Using $E = -\vec{m}_2\cdot\vec{B}_1$: If a South pole of magnet 2 (where field lines enter) is placed in the field $\vec{B}_1$ from the North pole of magnet 1 (where field lines exit), the field $\vec{B}_1$ and the effective moment $\vec{m}_2$ (pointing towards the North pole of magnet 2) are roughly anti-aligned near the poles. $E = -\vec{m}_2\cdot\vec{B}_1$. If $\vec{m}_2$ points opposite to $\vec{B}_1$, the dot product is negative, making the energy E negative. Minimizing energy means making E as negative as possible, which happens when the distance is smallest. Thus, opposite poles attract.

Problem 15.4

Estimate the saturation magnetization for iron at 0 K.

Solution: Saturation magnetization M_S occurs when all atomic magnetic moments in the material are aligned parallel to the external field (page 239). It is the maximum possible magnetic moment per unit volume.

$$M_S = n \times m_{atom}$$

where n is the number density of atoms and m_{atom} is the magnetic moment per atom.

For iron (Fe):

- Crystal structure: Body-Centered Cubic (BCC)
- Lattice constant: $a = 0.287\,\text{nm} = 2.87 \times 10^{-10}$ m (standard value)
- Atoms per unit cell: 2
- Volume of unit cell: $V_{cell} = a^3 = (2.87 \times 10^{-10}\,\text{m})^3 \approx 2.36 \times 10^{-29}\,\text{m}^3$
- Number density of atoms: $n = \frac{\text{atoms}}{\text{cell}}/V_{cell} = 2/(2.36 \times 10^{-29}\,\text{m}^3) \approx 8.47 \times 10^{28}\,\text{atoms/m}^3$
- Magnetic moment per Fe atom: $m_{atom} \approx 2.2\mu_B$ (standard experimental value at 0 K, related to unpaired electron spins). $\mu_B = 9.274 \times 10^{-24}$ J/T is the Bohr magneton (page 235, Eq. 14.19).
- $m_{atom} = 2.2 \times (9.274 \times 10^{-24}\,\text{J/T}) \approx 2.04 \times 10^{-23}\,\text{J/T}$ (or A·m^2)

Now, calculate the saturation magnetization:

$$M_S = n \times m_{atom} = (8.47 \times 10^{28}\,\text{m}^{-3}) \times (2.04 \times 10^{-23}\,\text{A} \cdot \text{m}^2)$$

$$M_S \approx 1.73 \times 10^6\,\text{A/m}$$

The experimental value for saturation magnetization of iron at low temperatures is indeed around 1.7×10^6 A/m.

Problem 15.5

(a) Show that the area enclosed in a hysteresis loop in the (B,H) plane is equal to the energy dissipated in going around the loop. (b) Estimate the power dissipated if 1 kg of iron is cycled through a hysteresis loop at 60 Hz; the coercivity of iron is $4 \times 10^3 A/m$.

Solution: (a) The incremental work done per unit volume by an external source to change the magnetic state of a material by $d\vec{B}$ in the presence of a field $\vec{H}$ is $dW = \vec{H} \cdot d\vec{B}$. This is the energy supplied to the system per unit volume. Over a full cycle of the hysteresis loop, the net work done per unit volume is the integral around the closed loop:

$$W_{cycle} = \oint \vec{H} \cdot d\vec{B}$$

If we consider B and H to be scalar quantities aligned in the same direction, as is typical for a hysteresis loop measurement:

$$W_{cycle} = \oint H\,dB$$

This integral represents the area enclosed by the hysteresis loop in the B versus H plane. Since the material returns to its initial state after one cycle, this net work done on the system must be dissipated as heat, representing the energy loss due to hysteresis.

(b) The power dissipated is the energy lost per cycle multiplied by the frequency f.

$$P = W_{cycle,total} \times f = (W_{cycle,per\_volume} \times \text{Volume}) \times f$$

The energy loss per unit volume per cycle is the area of the B-H loop ($W_{cycle,per\_volume} = \oint H\,dB$). We need to estimate this area. A rough estimate of the area of the hysteresis loop is $Area \approx 4B_R H_C$, where B_R is the remanent flux density and H_C is the coercivity. Given $H_C = 4 \times 10^3$ A/m. We need B_R. $B_R = \mu_0(H_R + M_R)$. At $H = 0$, $B_R \approx \mu_0 M_R$. The remanent magnetization M_R is typically a significant fraction of the saturation magnetization M_S. Let's assume $M_R \approx 0.8 M_S$. From Problem 15.4, $M_S \approx 1.73 \times 10^6$ A/m. $M_R \approx 0.8 \times (1.73 \times 10^6\,\text{A/m}) \approx 1.38 \times 10^6$ A/m. $B_R \approx \mu_0 M_R = (4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A}) \times (1.38 \times 10^6\,\text{A/m}) \approx 1.74$ T.

Area estimate:

$$W_{cycle,per\_volume} \approx 4B_R H_C = 4 \times (1.74\,\text{T}) \times (4 \times 10^3\,\text{A/m}) \approx 27840\,\text{J/m}^3$$

(Note: $T \cdot A/m = (N/(A \cdot m)) \cdot (A/m) = N/m^2 = Pa = J/m^3$)

Now, find the volume of 1 kg of iron. Density of iron $\rho_{Fe} = 7874$ kg/m^3.

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}} = \frac{1\,\text{kg}}{7874\,\text{kg/m}^3} \approx 1.27 \times 10^{-4}\,\text{m}^3$$

Total energy dissipated per cycle:

$$W_{cycle,total} = (27840\,\text{J/m}^3) \times (1.27 \times 10^{-4}\,\text{m}^3) \approx 3.54\,\text{J}$$

Power dissipated at $f = 60$ Hz:

$$P = W_{cycle,total} \times f = (3.54\,\text{J}) \times (60\,\text{s}^{-1}) \approx 212\,\text{W}$$

So, about 212 Watts are dissipated as heat.

Problem 15.6

Approximately what current would be required in a straight wire to be able to erase a $\gamma - Fe_2O_3$ recording at a distance of 1 cm?

Solution: To erase a magnetic recording, the applied magnetic field H must be at least equal to the coercivity H_C of the magnetic medium. For gamma ferric oxide ($\gamma - Fe_2O_3$), the coercivity is $H_C = 300$ Oe (page 241). First, convert Oersteds (Oe) to A/m using the conversion factor from Equation (14.4) (page 232): $1\frac{A}{m} = \frac{4\pi}{1000}$ Oe.

$$H_C = 300\,\text{Oe} \times \frac{1\,\text{A/m}}{(4\pi/1000)\,\text{Oe}} = \frac{300 \times 1000}{4\pi}\,\text{A/m} \approx 23870\,\text{A/m}$$

The magnetic field strength H at a radial distance r from a long straight wire carrying current I is given by Ampere's Law:

$$H = \frac{I}{2\pi r}$$

We need to find the current I such that $H \geq H_C$ at $r = 1\,\text{cm} = 0.01$ m.

$$\frac{I}{2\pi r} = H_C$$

$$I = 2\pi r H_C$$

$$I = 2\pi \times (0.01\,\text{m}) \times (23870\,\text{A/m})$$

$$I \approx 1500\,\text{A}$$

A current of approximately 1500 Amperes would be required in a straight wire to generate a field equal to the coercivity of $\gamma - Fe_2O_3$ at a distance of 1 cm. This is a very large current, highlighting that specialized recording heads producing focused fields are necessary for magnetic recording and erasure.